J Glob Optim (2008) 40:197–208
DOI 10.1007/s10898-007-9187-4

Constrained extremum problems with infinite dimensional image. Selection and saddle point

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Received: 29 May 2007 / Accepted: 29 May 2007 / Published online: 6 July 2007

Abstract The paper deals with Image Space Analysis for constrained extremum problems having infinite dimensional image. It is shown that the introduction of selection for point-to-set maps and of quasi-multipliers allows one to establish sufficient optimality conditions for problems, where the classic ones fail.

Keywords Optimality conditions · Saddle point · Multipliers · Quasi-multipliers · Image Space Analysis

Mathematics Subject Classification (2000) 90C · 65K

1 Introduction

Assume we are given the integers m, n and p with $m \geq 0$, $0 \leq p \leq m$, $n > 0$, the interval $T := [a, b] \subset \mathbb{R}$ with $-\infty \leq a \leq b \leq +\infty$, and the functions

$$\psi_i : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad i = 0, 1, \ldots, m.$$

Let $\mathbb{V}$ be a subset of $C^0(T)^n$, namely, the set of all continuous functions $x(t) = (x_1(t), \ldots, x_n(t))$ having continuous derivatives $x'(t) = (x_1'(t), \ldots, x_n'(t))$, $t \in T$, except at most for a finite number of points $\bar{t}$ at which exist and are finite $\lim_{t\downarrow\bar{t}} x'(t)$ and $\lim_{t\uparrow\bar{t}} x'(t)$; $x'(\bar{t}) := \lim_{t\downarrow\bar{t}} x'(t)$. $\mathbb{V}$ forms a vector space on the set of real numbers, and is equipped with the norm

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$||x||_\infty := \max_{t \in T} ||x(t)||,\ x \in \mathbb{V}$, where $||\bullet||$ denotes the Euclidean norm in $\mathbb{R}^n$. X is defined as the subset of $\mathbb{V}$, whose elements satisfy a boundary condition, like fixed endpoint condition $x(a) = x^0$ and $x(b) = x^1$, x^0 and x^1 being given vectors of $\mathbb{R}^n$.

Let us consider the following geodesic-type minimization problem:

$$f^{\downarrow} := \min[f(x) := \int_T \psi_0(t, x(t), x'(t))\mathrm{d}t], \tag{1a}$$

subject to

$$\psi_i(t, x(t), x'(t)) = 0, \quad \forall t \in T,\ i \in \mathcal{I}^0 := \{1, \ldots, p\}, \tag{1b}$$

$$\psi_i(t, x(t), x'(t)) \geq 0, \quad \forall t \in T,\ i \in \mathcal{I}^+ := \{p+1, \ldots, m\}, \tag{1c}$$

$$x \in X \subseteq C^0(T)^n, \tag{1d}$$

where $p = 0 \Rightarrow \mathcal{I}^0 = \varnothing$, $p = m \Rightarrow \mathcal{I}^+ = \varnothing$, $m = 0 \Rightarrow \mathcal{I} := \mathcal{I}^0 \cup \mathcal{I}^+ = \varnothing$. Unless differently stated, we will assume that card $X > 1$.

Let us set $D := O_p \times \mathbb{R}_+^{m-p}$ with $O_p := (0, \ldots, 0) \in \mathbb{R}^p$; we stipulate that $D = \mathbb{R}_+^m$ when $p = 0$ and $D = O_m := (0, \ldots, 0) \in \mathbb{R}^m$ when $p = m$; $m = 0$ does not require to define D. Set $\psi := (\psi_1, \ldots, \psi_m)$. The set

$$R := \{x \in X : \psi(t, x(t), x'(t)) \in D, \quad \forall t \in T\}$$

is the feasible region of problem (1).

In the image space approach [2], the optimality of a feasible point is expressed by means of the disjunction of suitable subsets $\mathcal{K}$ and $\mathcal{H}$ of the image space defined as the product space where the images of the objective and the constraint functions run.

The disjunction between $\mathcal{K}$ and $\mathcal{H}$ is proved by showing that they lie in two disjoint level sets of a separating functional. When such a functional can be found linear, we say that $\mathcal{K}$ and $\mathcal{H}$ admit a linear separation.

A key point in the analysis is represented by the dimension of the image space, which can be finite or infinite, according to the nature of the constraints. For example, the image associated to an isoperimetric problem is finite-dimensional, while the geodesic-type problem (1) has an infinite-dimensional image.

The case of a finite-dimensional image has been widely studied and many of the results obtained in this context can be generalized to an infinite dimensional problem under suitable additional assumptions (in particular, that $\mathcal{K}$ or $\mathcal{H}$ has non-empty interior).

A further possibility of developing the analysis consists in associating with (1) a finite-dimensional image. This can be achieved by considering the constraints as multifunctions with values in suitable subsets of a finite-dimensional space.

In the hypothesis of continuity of the functions involved, the existence of a selection for the image multifunction, such that its range has an empty intersection with a suitable subset of the image space, is a necessary and sufficient optimality condition for (1).

Such a selection can be expressed as a weighted integral and the weights, called "selection quasi multipliers", can be considered an enlargement of the class of multipliers associated with (1). When the selection multipliers do not locally depend on the unknown x of the problem, the classic necessary or sufficient optimality conditions of Calculus of Variations can be recovered.

Necessary optimality conditions have been analysed in [5], while in the present paper we focus our attention on sufficient optimality conditions. In particular, we will provide some examples where, by considering the class of selection quasi- multipliers, it is possible to obtain saddle point conditions for a generalized Lagrangian associated with (1).

The analysis carried out in the paper can be performed also locally by replacing X with $X \cap N_\rho(\bar{x})$, where $N_\rho(\bar{x})$ is the ball of center $\bar{x}$ and radius $\rho > 0$.

2 General features of image space analysis

Consider any $\overline{x} \in R$ and observe that $\overline{x}$ is a global minimum point (1), iff the system (in the unknown x):

$$f(\overline{x}) - f(x) > 0, \quad \psi(t, x(t), x'(t)) \in D, \quad \forall t \in T, \ x \in X \tag{2}$$

is impossible.

When $x \in X$ is fixed, then ψ becomes a function of t only. The set of the functions $\tilde{\psi}(x), x \in X$ where $\tilde{\psi}(x)(t) = \psi(t, x(t), x'(t))$, is a subset of an infinite dimensional space. Therefore, unlike what happens for isoperimetric-type problems (and, of course, for extremum problems in $\mathbb{R}^n$) the analysis of the image of (1) or (2) should be carried on in a Banach Space. Such an infinite dimensionality cannot be deleted; however, it can be postponed to the introduction of the Image Space (IS). This can be done by the following approach.

The image of x through $\tilde{\psi}_i$ is again a function defined on T; the image of $\tilde{\psi}_i(x)(\cdot)$ is a subset of $\mathbb{R}$. Hence, we can introduce the multifunction, which sends x into a subset of $\mathbb{R}^{1+m}$, namely $A_{\overline{x}} : X \rightrightarrows Y \subseteq \mathbb{R}^{1+m}$, defined by:

$$\begin{aligned} A_{\overline{x}}(x) : = \{(u, v) \in \mathbb{R}^{1+m} : u = f(\overline{x}) - f(x) \text{ and } \exists\, t \in T \text{ s.t.} \\ v_i = \psi_i(t, x(t), x'(t)), i \in \mathcal{I}\}. \end{aligned}$$

$$A_{\overline{x}}(x) := \{(u, v) \in \mathbb{R}^{1+m} : u = f(\overline{x}) - f(x), \ v_i = \psi_i(t, x(t), x'(t)), t \in T, i \in \mathcal{I}\}.$$

$\mathcal{K}_{\bar{x}} := A_{\overline{x}}(X)$ is called the *image* of (1). By means of the above multifunction, we are able to work in a finite dimensional Image Space, namely $\mathbb{R}^{1+m}$; the infinite dimensionality has not been deleted, but postponed, and it will appear again later in terms of selection from $A_{\overline{x}}(X)$.

By introducing the set $\mathcal{H} := (\mathbb{R}_+ \backslash \{O\}) \times D$, it is easy to see that (2) is impossible, iff

$$A_{\overline{x}}(x) \nsubseteq \mathcal{H}, \ \forall x \in X. \tag{3}$$

The infinite dimensionality, which has been postponed in order to be able to introduce a finite dimensional IS, appears now with the selection, $\forall x \in X$, of an element of $A_{\overline{x}}(x)$.

Consider the functions $\omega_i : T \to \mathbb{R}, i \in \mathcal{I}$. Denote by Ω the set of vectors $\omega := (\omega_1, \ldots, \omega_m)$, whose elements are not all identically zero on T and such that $\omega_i \geq 0, i \in \mathcal{I}^+$; Ω represents a class of functional parameters satisfying a suitable condition, under which the integral in (4) makes sense. The selection in this case is specified to be of type $\Phi : 2^{\mathbb{R}^{1+m}} \times \Omega \to \mathbb{R}^{1+m}$, defined by:

$$\Phi(A_{\bar{x}}(x), \omega) := \int\limits_{A_{\overline{x}}(x)} \omega \mathrm{d}t = \left(f(\overline{x}) - f(x), \ \int\limits_T \omega_i(t)\psi_i(t, x(t), x'(t))\mathrm{d}t, \ i \in \mathcal{I} \right) \tag{4}$$

where the 1st integral is a short writing to mean selection of an element of $A_{\overline{x}}(x)$ by means of a weighted integration.

Definition 1 Φ is called *generalized selection function* of $A_{\overline{x}}$ (for short, GSF), iff

$$\forall x \in X, \quad A_{\overline{x}}(x) \subseteq \mathcal{H} \Leftrightarrow \Phi(A_{\overline{x}}, \omega) \in \mathcal{H}, \ \forall \omega \in \Omega. \tag{5}$$

Observe that (5) is equivalent to (note that ω depends on x):

$$\forall x \in X, \quad A_{\overline{x}}(x) \nsubseteq \mathcal{H} \Leftrightarrow \exists \omega \in \Omega \quad s.t. \quad \Phi(A_{\overline{x}}(x), \omega) \notin \mathcal{H}. \tag{6}$$

ω is called *selection multiplier* (for short, SM).

From the so-called Fundamental Lemma of the Calculus of Variations [6], we draw the following:

Lemma 1 *Let* $\alpha \in C^0[a, b]$ *be such that*:

$$\int_a^b \alpha(t)\phi(t)\mathrm{d}t \geq 0, \quad \forall \phi \in C_0^1[a, b],$$

where $C_0^1[a, b] := \{\phi \in C^1[a, b] : \phi(t) \geq 0, \forall t \in [a, b], \phi(a) = \phi(b) = 0\}$; *then*

$$\alpha(t) \geq 0, \quad \forall t \in [a, b].$$

Proof Ab absurdo, assume that $\exists \bar{t} \in [a, b]$ such that $\alpha(\bar{t})<0$; since α is continuous on $[a, b]$, then there exists an interval $[t_1, t_2] \subseteq [a, b]$ such that $\bar{t} \in [t_1, t_2]$ and

$$\alpha(t) < 0, \quad \forall t \in [t_1, t_2].$$

Choose $\bar{\phi} \in C_0^1[a, b]$ in the following way

$$\bar{\phi}(t) = \begin{cases} (t - t_1)^2(t - t_2)^2 & \text{if } t \in [t_1, t_2] \\ 0 & \text{if } t \in [a, b] \setminus [t_1, t_2]. \end{cases}$$

Then $\int_a^b \alpha(t)\bar{\phi}(t)\mathrm{d}t < 0$, a contradiction. □

Next theorem shows that under continuity hypotheses on the involved functions, (4) is a GSF.

Theorem 1 *If* $\Omega = C^0(T)^m$ *and* ψ_i, $i \in \mathcal{I}$ *are continuous, then* (4) *is a GSF.*

Proof Let $x \in X$. Suppose that $A_{\overline{x}}(x) \subseteq \mathcal{H}$, i.e. (2) holds.

$$f(\overline{x}) - f(x) > 0, \quad \psi_i(t, x(t), x'(t)) = 0, \quad i \in \mathcal{I}^0, \qquad \psi_i(t, x(t), x'(t)) \geq 0, \quad i \in \mathcal{I}^+, \quad \forall t \in T \tag{7}$$

Then, since $\omega_i \geq 0$, $i \in \mathcal{I}^+$, $\forall \omega \in \Omega$, we have

$$f(\overline{x}) - f(x) > 0, \int_T \omega_i(t)\psi_i(t, x(t), x'(t))\mathrm{d}t = 0, \ i \in \mathcal{I}^0,$$

$$\int_T \omega_i(t)\psi_i(t, x(t), x'(t))\mathrm{d}t \geq 0, \ i \in \mathcal{I}^+, \ \forall t \in T, \tag{8}$$

and hence, $\Phi(A_{\overline{x}}, \omega) \in \mathcal{H}$, $\forall \omega \in \Omega$. Conversely, assume that $\Phi(A_{\overline{x}}, \omega) \in \mathcal{H}$, $\forall \omega \in \Omega$, that is (8) holds whatever $\omega \in \Omega$ may be. Since $x \in \mathbb{V}$, then $\psi_i(t, x(t), x'(t))$ is a bounded function continuous on T, except for a finite number of points $t_1, \ldots, t_k$, $i \in \mathcal{I}$. Suppose that $i \in \mathcal{I}^0$. Applying the Fundamental Lemma of Calculus of Variations in the interval $[a, b] := [t_j, t_{j+1}]$, $j = 1, \dots, k-1$,we obtain

$$\psi_i(t, x(t), x'(t)) = 0, \quad t \in T, \quad i \in \mathcal{I}^0.$$

Analogously, for $i \in \mathcal{I}^+$, applying Lemma 1 with $\alpha(t) := \psi_i(t, x(t), x'(t)$, $\phi(t) := \omega(t)$, $t \in T$, $[a, b] := [t_j, t_{j+1}]$, $j = 1, \dots, k-1$, we obtain that

$$\psi_i(t, x(t), x'(t)) \geq 0, \quad t \in T, \quad i \in \mathcal{I}^+. \qquad \square$$

In the subsequent part of the paper we will assume that the hypotheses of Theorem 1 are fulfilled. The previous result leads us to introduce the selected problem.

Definition 2 Let $\omega(\cdot, x) \in \Omega$, $x \in X$; the following problem:

$$\min f(x) := \int_T \psi_0(t, x(t), x'(t))\mathrm{d}t, \tag{9a}$$

subject to

$$g_i(x, \omega_i) := \int_T \omega_i(t, x)\psi_i(t, x(t), x'(t))\mathrm{d}t = 0, \quad i \in \mathcal{I}^\circ \tag{9b}$$

$$g_i(x, \omega_i) := \int_T \omega_i(t, x)\psi_i(t, x(t), x'(t))\mathrm{d}t \geq 0, \quad i \in \mathcal{I}^+, \tag{9c}$$

$$x \in X, \tag{9d}$$

is called the *selected problem* associated with (1).

Set $g(x, \omega) := (g_1(x, \omega_1), \dots, g_m(x, \omega_m))$.

Definition 3 The set:

$$\mathcal{K}_{\overline{x}}(\omega) := \{(u, v) \in \mathbb{R} \times \mathbb{R}^m : u = f(\overline{x}) - f(x), v_i = g_i(x, \omega_i), i \in \mathcal{I}, x \in X\}$$

is called the *selected image* of problem (1).

Proposition 1 *$\overline{x} \in R$ is a (global) minimum point of problem (1), if and only if there exists a function $\overline{\omega}(t, x)$, $\overline{\omega}(\cdot, x) \in \Omega$, $x \in X$, such that:*

$$\mathcal{H} \cap \mathcal{K}_{\overline{x}}(\overline{\omega}) = \emptyset. \tag{10}$$

Proof (10) is equivalent to $\Phi(x; \bar{\omega}) \notin \mathcal{H}$, $\forall x \in X$, where Φ is defined by (4). Therefore, because of (6), we have that (10) is equivalent to (3). Hence, $\overline{x}$ is a global minimum point of (1) iff (10) holds. $\square$

Remark 1 Since it is known from the IS Analysis in the finite dimensional case that (10) is equivalent to

$$\mathcal{H} \cap (\mathcal{K}_{\overline{x}}(\overline{\omega}) - \mathrm{cl}\, \mathcal{H}) = \emptyset, \tag{11}$$

Proposition 1 can be equivalently written using (11) instead of (10).

The above analysis leads in a natural way to define, in the IS, a problem equivalent to (1). The problem

$$\max(u), \quad \text{s.t.} \quad (u, v) \in A_{\bar{x}}(x) \subseteq \mathbb{R} \times D, \; x \in X, \tag{12}$$

will be called *image problem*. When $A_{\bar{x}}(x)$ is single-valued, it collapses to (see Ref. [2]):

$$\max(u), \;\; \text{s.t.} \;\; (u, v) \in \mathcal{K}_{\bar{x}}, \;\; v \in D.$$

Note that the unknown argument of the image problem is a set, in particular a point if $A_{\bar{x}}(x)$ is single-valued.

Proposition 2 *$\tilde{x}$ is a solution of (1) iff $\tilde{u} := f(\overline{x}) - f(\tilde{x})$ is a solution of (12).*

Proof Let $\tilde{x}$ be a solution of problem (1), so that:

$$f(\tilde{x}) \leq f(x), \; \forall x \in X \text{ s.t. } \psi(t, x(t), x'(t)) \in D, \; \forall t \in T. \tag{13}$$

(13) can be equivalently written as $\tilde{u} \geq u$, $\forall (u, v) \in A_{\bar{x}}(x)$ such that $A_{\bar{x}}(x) \subseteq \mathbb{R} \times D$, $x \in X$, or

$$\tilde{u} = \max_{(u,v) \in A_{\bar{x}}(x) \subseteq \mathbb{R} \times D, x \in X} u.$$

Vice versa, from a solution $\tilde{u}$ of (12) we can easily recover (13). □

The set $\mathcal{K}_{\overline{x}}(\omega)$ plays for (1) the same role played by the image set for problems having finite dimensional image [1]. Therefore, it is conceivable to expect to extend to (1) the IS Analysis, which has been exploited for isoperimetric-type problems and those with real unknowns.

Proposition 3 *Let $\bar{x} \in R$. Then $\bar{x}$ is a global minimum point for (1), if and only if*

$$conv\; A_{\bar{x}}(x) \not\subseteq \mathcal{H}, \;\; \forall x \in X.$$

Proof Recall that $\mathcal{H}$ is convex and that a point $\bar{x}$ is a global minimum point for (1) iff $A_{\bar{x}}(x) \not\subseteq \mathcal{H}, \forall x \in X$.

Only if Ab absurdo, suppose that $\exists \tilde{x} \in X$, such that conv$A_{\bar{x}}(\tilde{x}) \subseteq \mathcal{H}$. This implies $A_{\bar{x}}(\tilde{x}) \subseteq \mathcal{H}$ and the optimality of $\bar{x}$ is contradicted.

If Ab absurdo, suppose that $\exists \hat{x} \in X$ such that $A_{\bar{x}}(\hat{x}) \subseteq \mathcal{H}$. It follows that conv$A_{\bar{x}}(\hat{x}) \subseteq \mathcal{H}$, which contradicts the assumption. □

3 A saddle point condition

The introduction of the selected problem (9) allows us to recover classic sufficient optimality conditions under the additional assumption that the selection multiplier ω depends only on t, namely

$$\omega(t, x) = \omega(t) \in \Omega, \quad \forall x \in X. \tag{14}$$

In such case, problem (9) is associated with the Lagrangian function $L : X \times D^* \times \Omega$ defined by

$$L(x; \lambda, \omega) := \int_T (\psi_0(t, x(t), x'(t)) - \sum_{i=1}^{m} \lambda_i \omega_i(t) \psi_i(t, x(t), x'(t)) \mathrm{d}t.$$

Let D^* denote the positive polar of D, $\Pi := \{\pi := (\theta, \lambda) \in \mathbb{R}_+ \times D^*\}$ be a set of parameters, and $\mathcal{W}(\Pi)$ denote the set of (linear) functions $w : \mathbb{R} \times \mathbb{R}^m \times \Pi \to \mathbb{R}$ defined by:

$$w(u, v; \theta, \lambda) = \theta u + \langle \lambda, v \rangle, \quad \theta \in \mathbb{R}_+, \ \lambda \in D^*. \tag{15}$$

It is easy to show that:

$$\mathcal{H} \subset \text{lev}_{>0}\, w(\bullet; \theta, \lambda), \ \forall \theta > 0, \ \forall \lambda \in D^*. \tag{16}$$

Proposition 4 *Let $\overline{x} \in R$. If there exist $\overline{\omega} \in \Omega, \overline{\theta} \in \mathbb{R}_+ \backslash \{O\}$ and $\overline{\lambda} \in D^*$, such that*:

$$\overline{\theta}[f(\overline{x}) - f(x)] + \sum_{i=1}^{m} \overline{\lambda}_i \int_T \overline{\omega}_i(t) \psi_i(t, x(t), x'(t))\mathrm{d}t \le 0, \ \forall x \in X, \tag{17}$$

then $\overline{x}$ is a global minimum point of (1)*, and we have*:

$$\overline{\lambda}_i \int_T \overline{\omega}_i(t) \psi_i(t, \overline{x}(t), \overline{x}'(t))\mathrm{d}t = 0, \ \forall i \in \mathcal{I}. \tag{18}$$

Proof Taking into account Definitions 2 and 3, (17) implies that:

$$\mathcal{K}_{\overline{x}}(\overline{\omega}) \subseteq \text{lev}_{\le 0}\, w(\bullet; \overline{\theta}, \overline{\lambda}).$$

This inclusion, (16) for $(\theta, \lambda) = (\overline{\theta}, \overline{\lambda})$, and Proposition 1 prove the 1st part of the thesis. Since $\overline{x} \in R$, then (18) is trivial, for $i \in \overset{\circ}{\mathcal{I}}$. (1c), (5) and (17) for $x = \overline{x}$, and $\overline{\lambda}_i \ge 0, \ i \in \mathcal{I}^+$ (implied by $\overline{\lambda} \in D^*$) let us achieve (18) also for any $i \in \mathcal{I}^+$. □

Following the assumption of the above proposition, we assume $\theta > 0$ so that it is not restrictive to assume $\theta = 1$.

Theorem 2 *Let $\Omega := C^0(T)^m$ and ψ_i be continuous, $i \in \mathcal{I}$. If there exist $\overline{\lambda} \in D^*$, $\overline{\omega} \in \Omega$ such that:*

$$L(\overline{x}; \lambda, \omega) \le L(\overline{x}; \overline{\lambda}, \overline{\omega}) \le L(x; \overline{\lambda}, \overline{\omega}), \ \forall x \in X, \ \forall (\lambda, \omega) \in D^* \times \Omega, \tag{19}$$

then $\overline{x}$ is a global minimum point of (1).

Proof Set $\overline{\psi}_i(t) := \psi_i(t, \overline{x}(t), \overline{x}'(t)), \ i \in \mathcal{I}$. The 1st of (19) is equivalent to:

$$\sum_{i \in \mathcal{I}} \int_T \lambda_i \omega_i(t) \overline{\psi}_i(t)\mathrm{d}t \ge \sum_{i \in \mathcal{I}} \int_T \overline{\lambda}_i \overline{\omega}_i(t) \overline{\psi}_i(t)\mathrm{d}t, \ \forall (\lambda, \omega) \in D^* \times \Omega. \tag{20}$$

Let us prove that $\overline{x} \in R$. Ab absurdo, suppose that $\exists r \in \mathcal{I}$ and $\overline{t} \in T$, such that either $\overline{\psi}_r(\overline{t}) \ne 0$ if $r \in \overset{\circ}{\mathcal{I}}$ or $\overline{\psi}_r(\overline{t}) < 0$ if $r \in \mathcal{I}^+$. Since $\overline{\psi}_r$ is continuous, then it is possible to find $\tilde{\omega}_r \in C^0(T)$ such that $\int_T \tilde{\omega}_r(t) \overline{\psi}_r(t)\mathrm{d}t$ has the same sign as $\overline{\psi}_r(\overline{t})$. Therefore, by setting $\tilde{\omega}_i \equiv 0, \lambda_i = \overline{\lambda}_i, \ i \in \mathcal{I} \backslash \{r\}$,,we have that $\tilde{\omega} := (0, \ldots, \tilde{\omega}_r, \ldots, 0) \in \Omega$ and letting λ_r go to either $+\infty$ or $-\infty$, according to, respectively, $\overline{\psi}_r(\overline{t}) < 0$ or $\overline{\psi}_r(\overline{t}) > 0$, the left-hand side of (20) goes to $-\infty$ and contradicts (20). Hence $\overline{x} \in R$. Next we prove that (18) is fulfilled. Since $\overline{x} \in R$ and $(\overline{\lambda}, \overline{\omega}) \in D^* \times \Omega$ then

$$\overline{\lambda}_i \int_T \overline{\omega}_i(t) \overline{\psi}_i(t)\mathrm{d}t \ge 0, \ \forall i \in \mathcal{I}. \tag{21}$$

Moreover, by setting $\lambda = 0$ in (20), we obtain that

$$\sum_{i \in \mathcal{I}} \overline{\lambda}_i \int_T \overline{\omega}_i(t)\overline{\psi}_i(t)\mathrm{d}t \leq 0. \tag{22}$$

By (21) we have that equality holds in (22), (18) follows.

Taking into account (18), the 2nd of (19) becomes:

$$\int_T \psi_0(t, x, x')\mathrm{d}t \geq \int_T \psi_0(t, \overline{x}, \overline{x}')\mathrm{d}t + \sum_{i \in \mathcal{I}} \int_T \overline{\lambda}_i \overline{\omega}_i(t)\psi_i(t, x, x')\mathrm{d}t, \ \forall x \in X,$$

so that the above inequality implies $f(x) \geq f(\overline{x})$ for each $x \in R$. □

Corollary 1 *Let $\Omega := C^0(T)^m$ and ψ_i be continuous, $i \in \mathcal{I}$. If there exist $\overline{\lambda} \in D^*$, $, \overline{\omega} \in \Omega$ such that (19) holds, then $(\overline{x}, \overline{\lambda})$ is a saddle point on $X \times D^*$ for the Lagrangian $L(x; \lambda, \overline{\omega})$ associated with (9) where $g_i(x, \omega_i) := g_i(x, \overline{\omega}_i)$, $i \in \mathcal{I}$, and, therefore, $\overline{x}$ is a global minimum point for (9).*

Theorem 3 *Let $\mathcal{H}_u := \{(u, v) \in \mathbb{R}^{1+m} : u > 0, v = 0\}$ and assume $\exists\, \bar{\omega} \in \Omega$ such that*

$$TC(conv\ (\mathcal{K}_{\bar{x}}(\bar{\omega}) - cl\ \mathcal{H})) \cap \mathcal{H}_u = \emptyset, \tag{23}$$

where TC denotes the (Bouligand) tangent cone. If $\bar{x} \in R$, then $\exists \bar{\lambda} \in D^*$ such that $(\overline{x}, \bar{\lambda}, \bar{\omega})$ is a saddle point of $L(x; \lambda, \omega)$ on $X \times (D^* \times \Omega)$ [i.e. (19) holds] and (18) is fulfilled.

Proof Let us set

$$f(x) = \int_T \psi_0(t, x(t), x'(t))\mathrm{d}t \text{ and } g_i(x, \bar{\omega}_i) = \int_T \bar{\omega}_i \psi_i(t, x(t), x'(t)\mathrm{d}t, \ i \in \mathcal{I}.$$

The condition (23) is equivalent to regular separation between $\mathcal{H}$ and $\mathcal{K}_{\bar{x}}(\bar{\omega})$ [3], i.e. to the existence of $\bar{\lambda} \in D^*$, such that $u + \langle \bar{\lambda}, v \rangle \leq 0, \quad \forall (u, v) \in \mathcal{K}_{\bar{x}}(\bar{\omega})$.

This can be rewritten as

$$f(\bar{x}) \leq f(x) - \langle \bar{\lambda}, g(x, \bar{\omega}) \rangle, \quad \forall x \in X. \tag{24}$$

Setting $x = \bar{x}$ in (24), we obtain $\langle \bar{\lambda}, g(\bar{x}, \bar{\omega}) \rangle = 0$. Substracting the last scalar product from the left-hand side of (24), we get:

$$f(\bar{x}) - \langle \bar{\lambda}, g(\bar{x}, \bar{\omega}) \rangle \leq f(x) - \langle \bar{\lambda}, g(x, \bar{\omega}) \rangle, \quad \forall x \in X. \tag{25}$$

On the other side, we have $\langle \lambda, g(\bar{x}, \omega) \rangle \geq 0, \forall (\lambda, \omega) \in D^* \times \Omega$, which implies:

$$f(\bar{x}) - \langle \lambda, g(\bar{x}, \omega) \rangle \leq f(\bar{x}) - \langle \lambda, g(\bar{x}, \bar{\omega}) \rangle \leq f(\bar{x}) - \langle \bar{\lambda}, g(\bar{x}, \bar{\omega}) \rangle, \ \forall (\lambda, \omega) \in D^* \times \Omega. \tag{26}$$

From (26) and (25) we achieve the thesis. □

Apart from the splitting of the classic Lagrange multiplier, say $\lambda_i(t)$, into λ_i and $\omega_i(t)$, Theorem 2 is a classic condition. The Image Space Analysis, with the above factorization of the classic multiplier, leads to see that an improvement is possible. Next examples show simple obstacle problems for which (19) is not fulfilled.

Example 1 In (1) set $T = [0, 1]$, $p = 0, m = 1$, $\psi_0(t, x, x') = x$, $\psi_1(t, x, x') = x^3(t)$, $\overline{x} \equiv 0$, $\forall t \in T$, $\mathbb{V} = C^0(T)$, $X = \{x \in \mathbb{V} : x(0) = x(1) = 0\}$. We have $\mathcal{H} = (\mathbb{R}_+ \backslash \{O\}) \times \mathbb{R}_+$.

We will show that such a problem does not admit a saddle point for the Lagrangian function (here $\lambda_1 = \lambda$, $\omega_1 = \omega$):

$$L(x; \lambda, \omega) = \int_T (x(t) - \lambda\omega(t)x^3(t))\mathrm{d}t,$$

where $\omega \in C^0(T)$. Here $\Omega := \{\omega \in C^0(T) : \omega \geq 0, \omega \not\equiv 0\}$. To this aim, we will show that $\sup_{\lambda \geq 0} \sup_{\omega \in \Omega} \inf_{x \in X} L(x; \lambda, \omega) = -\infty$ or, equivalently,

$$\inf_{x \in X} L(x; \lambda, \omega) = -\infty, \quad \forall \omega \in \Omega, \ \forall \lambda \geq 0. \tag{27}$$

Let $W(t) := \int_0^t \omega(s)\mathrm{d}s$ and $I := W(1)$. If $\lambda = 0$, then $\inf_{x \in X} L(x; \lambda, \omega) = -\infty$. Suppose that $\lambda \neq 0$. Since $\omega(t) \not\equiv 0$ and $\omega(t) \geq 0$, on T, then $W(t)$ is a non-identically zero and non-decreasing function with $W(0) = 0$. Let $\bar{t} \in T$ be s.t. $W(\bar{t}) = I/2$ and consider the function:

$$x_a(t) = \begin{cases} 0, & \text{if } \ 0 \leq t \leq \bar{t}, \\ -a \ \sin\left(\frac{2\pi}{I} W(t)\right), & \text{if } \ \bar{t} \leq t \leq 1, \end{cases}$$

with $a > 0$. $L(x_a; \lambda, \omega) =$

$$\int_T (x_a(t) - \lambda\omega(t)x_a^3(t))\mathrm{d}t = \int_{\bar{t}}^1 \left[-a \ \sin\left(\frac{2\pi}{I} W(t)\right) + a^3 \ \sin^3\left(\frac{2\pi}{I} W(t)\right)\omega(t)\lambda\right]\mathrm{d}t.$$

Putting $y = \frac{2\pi}{I} W(t)$ we get $\mathrm{d}y = \frac{2\pi}{I}\omega(t)\mathrm{d}t$ and we obtain

$$\int_{\bar{t}}^1 a^3 \ \sin^3\left(\frac{2\pi}{I} W(t)\right)\omega(t)\lambda\mathrm{d}t = \int_{\pi}^{2\pi} \frac{Ia^3}{2\pi} \ \sin^3(y)\lambda dy = -\frac{2a^3 I\lambda}{3\pi},$$

$$\text{while} \quad \int_{\bar{t}}^1 -a \ \sin\left(\frac{2\pi}{I} W(t)\right)\mathrm{d}t \leq a(1 - \bar{t}).$$

$$\text{Then} \quad L(x_a; \lambda, \omega) \leq -\frac{2a^3 I\lambda}{3\pi} + a(1 - \bar{t}) \to -\infty, \quad a \to +\infty$$

which implies (27). □

In the next example we show that, under the assumption (14), it is not always possible to find a SM such that (1) is equivalent to (9). By Corollary 1, in such cases (19) cannot be fulfilled.

Example 2 In (1) set $T = [0, 1]$, $p = 1, m = 2, n = 2$, $\psi_0(t, x(t), x'(t)) = -x_1^2(t) + x_2(t)$, $\psi_1(t, x(t), x'(t)) = x_1(t)$, $\psi_2(t, x(t), x'(t)) = x_2^3(t)$, $\overline{x}(t) = (0, 0)$, $\forall t \in T$, $\mathbb{V} = C^0(T) \times C^0(T)$, $X = \{x \in \mathbb{V} : x_i(0) = x_i(1) = 0, i = 1, 2\}$.We have $\mathcal{H} = (\mathbb{R}_+ \backslash \{O\}) \times \{O\} \times \mathbb{R}_+$. Consider problem (9) where we have supposed that the SM are independent of x, namely $\omega := (\omega_1, \omega_2) \in C^0(T)^2$. We want to show that (9) and (1) have different optimal values, whatever $\omega \in C^0(T)^2$ may be. We observe that this is true for $\omega_1 \equiv 0$. In such a case (9) is unbounded from below. Suppose that $\omega_1 \not\equiv 0$ and let $W(t) := \int_0^t \omega_1(s)\mathrm{d}s$, $I := W(1)$. Since $\omega_1 \not\equiv 0$, then there exists an open interval $]a, b[\subset T$, such that:

$$\omega_1(t) > 0 \quad \text{or} \quad \omega_1(t) < 0, \quad \forall t \in]a, b[. \tag{28}$$

Consider the function $\hat{x}(t) = (\hat{x}_1(t), \hat{x}_2(t))$ where: $\hat{x}_1(t) = \sin\left(\frac{2\pi}{I}W(t)\right)$, $\hat{x}_2(t) = 0$, $t \in T$. It is easy to show that $\hat{x}(t)$ is a feasible solution for (9):

$$\int_0^1 \sin\left(\frac{2\pi}{I}W(t)\right)\omega(t)\mathrm{d}t = \int_0^{2\pi} \frac{I}{2\pi}\sin(y)\mathrm{d}y = 0,$$

where we have put $y = \frac{2\pi}{I}W(t)$ which implies $dy = \frac{2\pi}{I}\omega(t)\mathrm{d}t$. Moreover

$$\int_T \psi_0(t, \hat{x}(t), \hat{x}'(t)\mathrm{d}t = -\int_0^1 \left[\sin^2\left(\frac{2\pi}{I}W(t)\right)\right]\mathrm{d}t \le -\int_a^b \left[\sin^2\left(\frac{2\pi}{I}W(t)\right)\right]\mathrm{d}t < 0.$$

The last inequality is due to the fact that, because of (28), the integrand function is strictly positive a.e. on $]a, b[$. This proves that the optimal value of (9) cannot coincide with the one of (1). □

4 Further developments

We have shown that well-known results, as Saddle Point Conditions, can be recovered assuming that the selection multipliers do not depend on x. Anyway, the examples of the previous section show that even elementary problems may escape from the classic Lagrange multipliers theory. In order to extend the validity of such a theory, here we consider a wider class of multipliers, called *quasi-multipliers,* which depend also on the unknown. This enlargement is suggested by the Image Space Analysis of Sect. 2, which has led to split the multiplier into two parts: selection from a multifunction, and separation of two sets. While the latter remains unchanged, the former can be extended. To this end, we now consider the functions $\omega_i : T \times X \to \mathbb{R}$, $i \in \mathcal{I}$, such that $\omega_i(\bullet, x) \in \Omega$, $\forall x \in X$ (see Definition 2.1). Without any fear of confusion, the domain of $\omega(t, x)$ is denoted again by Ω.

Next results are a straightforward generalization of those of Section 3.

Proposition 5 *Let $\overline{x} \in R$. If there exist $\overline{\omega}(t, x) \in \Omega$, $\overline{\theta} \in \mathbb{R}_+ \backslash \{O\}$ and $\overline{\lambda} \in D^*$, such that*:

$$\overline{\theta}[f(\overline{x}) - f(x)] + \sum_{i=1}^m \overline{\lambda}_i \int_T \overline{\omega}_i(t, x)\psi_i(t, x(t), x'(t))\mathrm{d}t \le 0, \ \forall x \in X, \tag{29}$$

then $\overline{x}$ is a global minimum point of (1)*, and we have*:

$$\overline{\lambda}_i \int_T \overline{\omega}_i(t, x)\psi_i(t, \overline{x}(t), \overline{x}'(t))\mathrm{d}t = 0, \ \forall i \in \mathcal{I}. \tag{30}$$

Proof Quite similar to that of Proposition 4. □

Assuming $\theta = 1$, now the selected problem is associated with the following extended Lagrangian function:

$$L(x; \lambda, \omega(t, x)) := \int_T (\psi_0 - \sum_{i=1}^m \lambda_i\, \omega_i(t, x)\psi_i)\mathrm{d}t,$$

which differs from that of Sect. 2 only because of the dependence of ω_i on x.

Theorem 4 *If there exist* $\overline{\lambda} \in D^*, \overline{\omega}(t,x) \in \Omega$, *such that:*

$$L(\overline{x};\lambda,\omega(t,\overline{x})) \leq L(\overline{x};\overline{\lambda},\overline{\omega}(t,\overline{x})) \leq L(x;\overline{\lambda},\overline{\omega}(t,x)), \quad \forall x \in X,\ \forall(\lambda,\omega) \in D^* \times \Omega, \tag{31}$$

then $\overline{x}$ *is a global minimum point of (1).*

Proof Quite similar to that of Theorem 2. □

As regards the examples analysed in Sect. 3, we will show that, using quasi-multipliers, we can overcome the presence of a positive duality gap.

Definition 4 The Lagrangian dual associated with (1) is defined by

$$\sup_{\lambda\in D^*} \sup_{\omega\in\Omega} \inf_{x\in X} L(x;\lambda,\omega).$$

$$v := f(\bar{x}) - \sup_{\lambda\in D^*} \sup_{\omega\in\Omega} \inf_{x\in X} L(x;\lambda,\omega)$$

is called the duality gap, provided that $\bar{x}$ is a global optimal solution to (1).

Continuation of Example 1 In Sect. 3, we have seen that, if we suppose that (14) holds then $v = +\infty$ (see 27). Assume now that $\omega = \omega(t,x)$. The Lagrangian dual is:

$$\sup_{\lambda\in D^*} \sup_{\omega\in\Omega} \inf_{x\in X} \int_0^1 (x(t) - \lambda\omega(t,x)x^3(t))\mathrm{d}t.$$

If we consider the function $\bar{\omega}_1(t,x) = \begin{cases} -x & \text{if } x \leq 0 \\ 0 & \text{if } x > 0 \end{cases}$, which belongs to Ω, then it is easy to show that:

$$\sup_{\lambda\in D^*} \inf_{x\in X} \int_0^1 (x(t) - \lambda\bar{\omega}_1(t,x)x^3(t))\mathrm{d}t = 0,$$

and thus the duality gap becomes 0. □

Continuation of Example 2 The Lagrangian associated with (1) is:

$$L(x_1,x_2;\lambda_1,\lambda_2,\omega_1,\omega_2) = \int_0^1 \left(-x_1^2 + x_2 - \lambda_1\omega_1x_1 - \lambda_2\omega_2x_2^3\right)\mathrm{d}t,$$

where $\omega_i = \omega_i(t,x_1,x_2)$, $x_i = x_i(t)$, $i = 1,2$.

Choose $\omega_1(t,x_1,x_2) = -x_1$ and $\omega_2(t,x_1,x_2) = \begin{cases} -x_2 & \text{if } x_2 \leq 0 \\ 0 & \text{if } x_2 > 0 \end{cases}$. Then $\omega = (\omega_1,\omega_2) \in \Omega$ and, being

$$\sup_{\lambda\in D^*} \inf_{x\in X} \int_0^1 \left(x_1^2(\lambda_1 - 1) + x_2 + \lambda_2 x_2^4\right)\mathrm{d}t = 0,$$

the duality gap becomes 0. □

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